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# NAMBU-LIE GROUPS ENDOWED WITH MULTIPLICATIVE TENSORS OF TOP ORDER

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**ABSTRACT.** A multiplicative (Nambu-Poisson) tensor of top order on a Lie group is characterized. As an application, we determine multiplicative structures on 3-dimensional Lie groups.

## 1. INTRODUCTION

A Nambu-Lie group is defined as a natural generalization of a Poisson Lie group. In fact, if  $\eta$  is a *multiplicative*  $k$ -vector field on a Lie group  $G$  which satisfies *fundamental identity*, then a pair  $(G, \eta)$  is called a *Nambu-Lie group*. If  $k = 2$ , then  $(G, \eta)$  is especially called a Poisson Lie group [1],[3]. A Nambu-Lie group was studied by J.Grabowski and G.Marmo [2] and I.Vaisman [5]. In [2], they proved that there are no Nambu-Lie structures of order  $k \geq 3$  on simple Lie groups. I.Vaisman [5] gave an alternative definition of multiplicativity by defining the  $k$ -bracket of 1-forms on  $G$ . In this paper, we characterize the properties of multiplicative Nambu-Poisson tensors of top order (*i.e.*,  $n = k$ ). Note that the word "Nambu-Poisson" is void in this case. As an application of these characterizations, we determine multiplicative (Nambu-Lie) structures defined on 3-dimensional Lie groups.

## 2. NAMBU-LIE GROUPS

Let  $G$  be an  $n$ -dimensional connected Lie group with the Lie algebra  $\mathfrak{g}$ . We denote by  $\Gamma(\Lambda^k TG)$  the set of  $k$ -vector fields (or contravariant tensor fields of order  $k$ ) on  $G$ . Let  $\mathcal{F}$  be the set of  $C^\infty$ -functions on  $G$ . Each element  $\eta$  of  $\Gamma(\Lambda^k TG)$  defines a  $k$ -bracket of functions  $f_i \in \mathcal{F}$  as follows.

$$\{f_1, \dots, f_k\} = \eta(df_1, \dots, df_k).$$

Since this  $k$ -bracket satisfies Leibniz rule, we can define a vector field  $X_{f_1, \dots, f_{k-1}}$  by

$$X_{f_1, \dots, f_{k-1}}(g) = \{f_1, \dots, f_{k-1}, g\}, \quad g \in \mathcal{F}.$$

**Definition 2.1.** An element  $\eta$  of  $\Gamma(\Lambda^k TG)$ ,  $k \geq 3$ , is called a Nambu-Poisson tensor of order  $k$  if  $\eta$  satisfies

$$\mathcal{L}_{X_{f_1, \dots, f_{k-1}}} \eta = 0$$

for any  $f_1, \dots, f_{k-1} \in \mathcal{F}$ .

Note that if  $k = n$ , every  $\eta$  is a Nambu-Poisson tensor [4].

**Definition 2.2.** An element  $\eta$  of  $\Gamma(\Lambda^k TG)$  is said to be multiplicative if  $\eta$  satisfies

$$\eta_{gh} = L_{g*} \eta_h + R_{h*} \eta_g$$

for any  $g, h \in G$ , where  $L_g$  and  $R_g$  denote, respectively, the left and the right translations. Let  $G$  be a Lie group endowed with a multiplicative Nambu-Poisson tensor  $\eta$ . Then a pair  $(G, \eta)$  is called a Nambu-Lie group.

For an element  $\Lambda \in \Lambda^k \mathfrak{g}$ , we define vector fields  $\bar{\Lambda}$  and  $\tilde{\Lambda}$  by

$$\bar{\Lambda}_g = L_{g*} \Lambda, \quad \tilde{\Lambda}_g = R_{g*} \Lambda, \quad \text{for all } g \in G.$$

Then it is clear that  $\bar{\Lambda}$  (resp.  $\tilde{\Lambda}$ ) is a left (resp. right) invariant vector field on  $G$ . Let us recall the following, which was proved by J-H Lu [3].

**Proposition 2.1.** Let  $G$  be a compact (or semisimple) Lie group. Then for every multiplicative  $k$ -vector field  $\eta \in \Gamma(\Lambda^k TG)$ , there exists an element  $\Lambda \in \Lambda^k \mathfrak{g}$  such that

$$\eta_g = \bar{\Lambda}_g - \tilde{\Lambda}_g$$

for all  $g \in G$ .

Using the above proposition, we show the following theorem.

**Theorem 2.2.** Let  $(G, \eta)$  be an  $n$ -dimensional compact or semisimple Nambu-Lie group, and let  $\eta$  be of top order. Then  $\eta = 0$ .

*Proof.* By Proposition 2.1, there exists an element  $\Lambda$  of  $\Lambda^n \mathfrak{g}$  such that  $\eta = \bar{\Lambda} - \tilde{\Lambda}$ . For all  $g, h \in G$ ,

$$Ad_g \bar{\Lambda}_h = R_{g^{-1}*} L_{g*} \bar{\Lambda}_h = R_{g^{-1}*} \bar{\Lambda}_{gh}.$$

On the other hand, since  $G$  is a unimodular Lie group, we have

$$Ad_g \bar{\Lambda}_h = (\det Ad_g) \bar{\Lambda}_{ghg^{-1}} = \bar{\Lambda}_{ghg^{-1}}.$$

Hence we obtain that  $R_{g^{-1}*} \bar{\Lambda}_{gh} = \bar{\Lambda}_{ghg^{-1}}$ . This means that a left invariant vector field  $\bar{\Lambda}$  is also a right invariant vector field. i.e.,  $R_{h*} \bar{\Lambda}_g = \bar{\Lambda}_{gh}$ . This equation induces

$$\begin{aligned} R_{h*} \bar{\Lambda}_g &= R_{h*} L_{g*} \Lambda = L_{g*} R_{h*} \Lambda \\ &= \bar{\Lambda}_{gh} = L_{g*} L_{h*} \Lambda. \end{aligned}$$

Thus we have  $R_{h*} \Lambda = L_{h*} \Lambda$  for all  $h \in G$ , and this means  $\eta = \bar{\Lambda} - \tilde{\Lambda} = 0$ .  $\square$

Let  $\eta$  be a Nambu-Poisson tensor of order  $k$  on  $G$ . Then  $\eta$  defines a bundle mapping

$$\sharp_\eta : \underbrace{T^*G \times \cdots \times T^*G}_{k-1 \text{ times}} \longrightarrow TG$$

given by

$$\langle \beta, \sharp_\eta(\alpha_1, \dots, \alpha_{k-1}) \rangle = \eta(\alpha_1, \dots, \alpha_{k-1}, \beta),$$

where all the arguments are covectors.

For such a tensor  $\eta$ , I.Vaisman [5] defined a  $k$ -bracket of 1-forms by

$$\{\alpha_1, \dots, \alpha_k\} = d(\eta(\alpha_1, \dots, \alpha_k)) + \sum_{j=1}^k (-1)^{k+j} i(\sharp_\eta(\alpha_1, \dots, \widehat{\alpha_j}, \dots, \alpha_k)) d\alpha_j,$$

where  $\alpha_j$  ( $j = 1, \dots, k$ ) are 1-forms on  $G$ .

The following theorem proved by I.Vaisman [5] gives one of the characterizations of Nambu-Lie groups.

**Theorem 2.3.** *If  $G$  is a connected Lie group endowed with a Nambu-Poisson tensor field  $\eta$  which vanishes at the unit  $e$  of  $G$ , then  $(G, \eta)$  is a Nambu-Lie group if and only if the  $k$ -bracket of any  $k$  left (right) invariant 1-forms of  $G$  is a left (right) invariant 1-form.*

Using Theorem 2.3, we characterize a multiplicative tensor  $\eta$  of top order. Let  $\mathfrak{g}$  be a Lie algebra of  $G$  with a basis  $X_1, \dots, X_n$ . We also denote the extended left invariant vector fields induced from  $X_i$  by the same letter. Since  $\eta$  is of top order,  $\eta$  has an expression  $\eta = fX_1 \wedge \cdots \wedge X_n$  for some  $f \in \mathcal{F}$ . Let  $\omega_i$  ( $i = 1, \dots, n$ ) be left invariant 1-forms dual to  $X_i$ . Under these notations we prove

**Theorem 2.4.** *Let  $\eta = fX_1 \wedge \cdots \wedge X_n$ ,  $f \in \mathcal{F}$  be a tensor of top order on  $G$ . (Recall that such a tensor is always a Nambu-Poisson tensor.) Then  $\eta$  is multiplicative if and only if  $f(e) = 0$  and*

$$X_i f + \left( \sum_{k=1}^n C_{ik}^k \right) f = q_i, \quad i = 1, \dots, n,$$

where  $\{C_{ij}^k\}$  are structure constants of  $\mathfrak{g}$  with respect to the basis  $X_1, \dots, X_n$ , and  $q_i$  ( $i = 1, \dots, n$ ) are some constants.

*Proof.* By Theorem 2.3, we know that  $\eta$  is multiplicative if and only if  $\eta_e = 0$  and

$$\begin{aligned} \{\omega_1, \dots, \omega_n\} &= d(\eta(\omega_1, \dots, \omega_n)) + \sum_{k=1}^n (-1)^{n+k} i(\sharp_\eta(\omega_1, \dots, \widehat{\omega_k}, \dots, \omega_n)) d\omega_k \\ &= df + f \sum_{k=1}^n i(X_k) d\omega_k = df + f \left( \sum_{\alpha, k=1}^n C_{\alpha k}^k \omega_\alpha \right) \end{aligned}$$

is a left invariant 1-form. Since  $\langle X_i, \{\omega_1, \dots, \omega_n\} \rangle$  is constant for any  $X_i$ , we have

$$\langle X_i, \{\omega_1, \dots, \omega_n\} \rangle = X_i f + \left( \sum_{k=1}^n C_{ik}^k \right) f = q_i, \quad i = 1, \dots, n.$$

□

### 3. EXAMPLES

In this section, as an application of Theorem 2.4, we calculate Nambu-Lie group structures (*i.e.*, multiplicative Nambu-Poisson tensors) of order 3 on 3-dimensional simply connected Lie groups. Since such tensors are of top degree, we have only to see whether they are multiplicative or not.

Throughout this section, we denote by  $G$  the simply connected Lie groups corresponding to Lie algebras  $\mathfrak{g}$ . Linearly independent three left invariant vector fields are denoted by  $X, Y, Z$ . Then  $\eta \in \Gamma(\Lambda^3 TG)$  is written as  $\eta = fX \wedge Y \wedge Z$ ,  $f \in C^\infty(G)$ . It is well-known that there are 9 types of 3-dimensional Lie algebras. If  $\mathfrak{g}$  is not a simple Lie algebra, its corresponding simply connected Lie group has global coordinates  $x, y, z$ . Hence a function  $f$  can be considered to be defined on  $\mathbb{R}^3(x, y, z)$ .

Type 1.  $[\mathfrak{g}, \mathfrak{g}] = 0$ . Namely  $\mathfrak{g}$  is an abelian Lie algebra. The corresponding Lie group  $G$  is given by

$$G = \left\{ \begin{pmatrix} e^x & 0 & 0 \\ 0 & e^y & 0 \\ 0 & 0 & e^z \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

Using these coordinates  $x, y, z$ , left invariant vector fields are written as  $X = \frac{\partial}{\partial x}$ ,  $Y = \frac{\partial}{\partial y}$ ,  $Z = \frac{\partial}{\partial z}$ . By Theorem 2.4, a function  $f(x, y, z)$  must satisfy  $f(0, 0, 0) = 0$ , and  $\frac{\partial f}{\partial x} = a$ ,  $\frac{\partial f}{\partial y} = b$ ,  $\frac{\partial f}{\partial z} = c$ , where  $a, b, c$  are some constants. Hence  $f = ax + by + cz$ , and

$$\eta = (ax + by + cz) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

gives a Nambu-Lie group structure on  $G$ .

By the similar method, we can get the results for other types.

Type 2.  $\dim[\mathfrak{g}, \mathfrak{g}] = 1$ . There are 2 cases as follows.

Case (1).  $\mathfrak{g} =$  Heisenberg Lie algebra.  $\mathfrak{g}$  is characterized by the condition  $[\mathfrak{g}, \mathfrak{g}] \subset 1$ -dimensional center. The corresponding Lie group  $G$  is given by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

A Nambu-Lie group structure on  $G$  is given by

$$\eta = (ax + by) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

Case (2). A Lie algebra  $\mathfrak{g}$  endowed with a property  $[\mathfrak{g}, \mathfrak{g}] \not\subset$  the center of  $\mathfrak{g}$ . The corresponding Lie group  $G$  is given by

$$G = \left\{ \begin{pmatrix} e^{y+z} & 0 & xe^y \\ 0 & e^y & 0 \\ 0 & 0 & e^y \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

$$\eta = \{ax + c(e^z - 1)\} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

gives a Nambu-Lie group structure on  $G$ .

Type 3.  $\dim[\mathfrak{g}, \mathfrak{g}] = 2$ .  $\mathfrak{g}^{(2)} = 0$ . There are 4 cases as follows.

Case (1). Left invariant vector fields  $X, Y, Z$  satisfy  $[X, Y] = 0$ ,  $[X, Z] = -X$ ,  $[Y, Z] = -X - Y$ . The corresponding Lie group  $G$  is given by

$$G = \left\{ \begin{pmatrix} e^{-z} & ze^{-z} & xe^{-2z} \\ 0 & e^{-z} & ye^{-2z} \\ 0 & 0 & e^{-2z} \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

We know that

$$\eta = c(e^{2x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

gives a Nambu-Lie group structure on  $G$ .

Case(2). Left invariant vector fields  $X, Y, Z$  satisfy  $[X, Y] = 0$ ,  $[X, Z] = -X$ ,  $[Y, Z] = -Y$ . The corresponding Lie group  $G$  is given by

$$G = \left\{ \begin{pmatrix} e^{-z} & 0 & xe^{-2z} \\ 0 & e^{-z} & ye^{-2z} \\ 0 & 0 & e^{-2z} \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

We have

$$\eta = c(e^{2x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

Case (3). Let  $\mathfrak{g}$  be a Lie algebra endowed with the following bracket relations.  $[X, Y] = 0$ ,  $[X, Z] = -X$ ,  $[Y, Z] = -qY$ , ( $q \neq 0, 1$ ). The corresponding Lie group  $G$  is given by

$$G = \left\{ \begin{pmatrix} e^{-qz} & 0 & xe^{-(q+1)z} \\ 0 & e^{-z} & ye^{-(q+1)z} \\ 0 & 0 & e^{-(q+1)z} \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

We have

$$\eta = c(e^{(q+1)x} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}.$$

Case (4). Let  $\mathfrak{g}$  be a Lie algebra endowed with the following bracket relations.  $[X, Y] = 0$ ,  $[X, Z] = -Y$ ,  $[Y, Z] = X - qY$ , ( $q^2 < 4$ ). The

corresponding Lie group  $G$  has rather complicated expression. Put  $k = q/2$ ,  $p = \sqrt{1 - k^2} = \sqrt{4 - q^2}/2$ . Then  $G$  is given by

$$G = \left\{ \begin{pmatrix} \frac{1}{p}e^{-kz}(-k \sin(pz) + p \cos(pz)) & -\frac{1}{p}e^{-kz} \sin(pz) & xe^{-2kz} \\ \frac{1}{p}e^{-kz} \sin(pz) & \frac{1}{p}e^{-kz}(p \cos(pz) + k \sin(pz)) & ye^{-2kz} \\ 0 & 0 & e^{-2kz} \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Then

$$\eta = \begin{cases} \frac{c}{q}(e^{qx} - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, & q \neq 0 \\ cx \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, & q = 0 \end{cases}$$

gives a Nambu-Lie group structure on  $G$ .

Type 4.  $\dim[\mathfrak{g}, \mathfrak{g}] = 3$ . It is well-known that such Lie algebras are simple, and there are 2 cases. The corresponding simply connected Lie groups are  $G_1 = SU(2)$  and  $G_2 = \widetilde{SL(2, \mathbb{R})}$ , where  $\widetilde{SL(2, \mathbb{R})}/\mathbb{Z} \cong SL(2, \mathbb{R})$ . Since  $G_1$  is compact, and  $G_2$  is semisimple, we have  $\eta = 0$  by Theorem 2.2.

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